Further Results on Order Statistics from the Generalized Log Logistic Distribution

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Abstract: Further to our earlier results, we derive exact explicit expressions for the triple and quadruple moments of order statistics from the generalized log-logistic distribution.

Key words: Order Statistics, Recurrence Relations, Generalized log-logistic distribution, triple and quadruple moments

INTRODUCTION

Recently Adeyemi[1], Adeyemi and Ojo[2] initiated the study into the recurrence relations for moments of order statistics from the generalized log-logistic distribution. We have obtained recurrence relations for single and product moments of order statistics from a symmetric, Adeyemi[3] and the generalized log-logistic distribution Adeyemi and Ojo[4].

In this paper, we present further results on our earlier studies by presenting recurrence relations for triple and quadruple moments of order statistics from the generalized log-logistic distribution.

The probability density function (pdf) of the GLL \(m_i\), \(m_j\) distribution is given by

\[
xF'(\theta) = \gamma [F(\theta)]^{m_i} [1 - F(\theta)]^{m_j}; \quad \gamma = \frac{\alpha}{B(\frac{m_i}{\sigma}, \frac{m_j}{\sigma})}
\]

Letting \(\alpha = 1\) and \(\beta = -\mu\sigma\ln\left(\frac{m_i}{m_j}\right)\) it can be easily shown that the pdf of \(m_i, m_j\) becomes

\[
f(x) = \frac{1}{\sigma B(\frac{m_i}{\sigma}, \frac{m_j}{\sigma})} \frac{m_i^{m_i - 1} m_j^{m_j - 1}}{(1 + \frac{m_i}{m_j}) (e^{-\frac{x}{\sigma}})^{m_i + m_j}}
\]

Note that if \(m_i - m_j = 1\), GLL\((m_i, m_j)\) becomes the log-logistic distribution. It is symmetric around \(\ln(1) = \frac{-\beta'}{\alpha}\) if \(m_i = m_j\), positive skewed if \(m_i > m_j\) and negative skewed if \(m_j > m_i\).

Let \(X_{1,2,3,\ldots,n}\) denote the order statistics obtained when the \(n\) \(X\)'s are arranged in increasing order of magnitude. We denote

\[
\mu^{(s,b,c,d)} = E[X_{1,s} X_{2,b} X_{3,c} X_{4,d}], \quad 1 \leq s < t \leq n
\]

and

\[
\mu^{(s,b,c,d)} = E[X_{r,s} X_{b,t} X_{c,u} X_{d,n}], \quad 1 \leq r < s < t < u \leq n
\]

Also

\[
f_{r,s,t,u,w}(w,x,y,z) = \frac{c_{r,s,t,u}}{c_r} \frac{F(w)^{r-1} F(x)^{t-1} F(y)^{u-1} F(z)^{w-1}}{[1 - F(w)]^{r-1} [1 - F(x)]^{t-1} [1 - F(y)]^{u-1} [1 - F(z)]^{w-1}}
\]

where

\[
c_{r,s,t,u} = \frac{n!}{(r-1)! (s-1)! (t-1)! (u-1)! (n-r-t-u)!}
\]

and

\[
f_{r,s,t,u,w}(w,x,y,z) = \frac{c_{r,s,t,u}}{c_r} \frac{F(w)^{r-1} F(x)^{t-1} F(y)^{u-1} F(z)^{w-1}}{[1 - F(w)]^{r-1} [1 - F(x)]^{t-1} [1 - F(y)]^{u-1} [1 - F(z)]^{w-1}}
\]

where

\[
\gamma_{r,s,t,u} = \frac{n!}{(r-1)! (s-1)! (t-1)! (u-1)! (n-r-t-u)!}
\]

Adeyemi[1] and Adeyemi and Ojo[2] have obtained recurrence relations for \(\mu^{(s,b,c,d)}\) and expressions for \(\mu_{r,s,t,u}\) in both symmetric and general cases respectively.

In this paper, we obtain recurrence relations for \(\mu^{(s,b,c,d)}\) and \(\mu_{r,s,t,u}\) for positive integers \(m_i, m_j\).

Recurrence relations for triple moments: Theorem 2.1 for \(1 \leq r < s < t \leq n - m_i - 1\) and \(a, b, c \geq 1\)

\[
A_1(i) \mu_{r+s+i+t+1}\mu_{s+t+i+1}\mu_{r+i+1}\mu_{r+i+1+n} = \frac{\alpha \gamma (r-1)! (n-t)!}{m_i}
\]

\[
A_2(i) \mu_{r+i+1}\mu_{s+i+1}\mu_{r+s+i+1}\mu_{r+s+i+1+n} = \frac{s-r-1}{m_i}
\]

\[
A_3(i) \mu_{r+i+1}\mu_{s+i+1}\mu_{r+s+i+1}\mu_{r+s+i+1+n} = \frac{r+m_i-1}{m_i}
\]

where

\[
A_{r,i} = \sum_{i=0}^{m_i-1} \frac{m_i-1}{i} (-1)^{i} (r+i+1)!(n-t-m_i-i)!
\]

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\[ A_s(i) = \sum_{i=0}^{m_2} \binom{m_2}{i} (-1)^{(r+m_1+i-1)!(n-t-m_1-i+1)!} \]

and

\[ A_t(i) = \sum_{i=0}^{m_2} \binom{m_2}{i} (-1)^{(r+m_1+i-2)!(n-t-m_1-i+1)!} \]

Proof

\[ \gamma^{(a, b, c)}_{s, t, n} = C_{r, s, t, n} \int_{x} \int_{y} \gamma x^b y^c [F(y) - F(x)]^{s-1} [1 - F(y)]^{t-1} I_s(x) f(x) f(y) \, dx \, dy \]  \hspace{1cm} (2.2)

having used (1.1), (1.3) and (1.5) where

\[ I_s(x) = \int_w \left[ w^s [F(w)]^{r+m_2} [1 - F(w)]^{m_2} [F(x) - F(w)]^{t-1} \right]^{s-1} \, dw \]  \hspace{1cm} (2.3)

Integrating by parts, we have

\[ I_s(x) = (s-r-1) \int_w w^s [F(w)]^{r+m_1-1} [1 - F(w)]^{m_2} [F(x) - F(w)]^{t-1} F(w) \, dw \]

\[ + m_2 \int_w w^s [F(w)]^{r+m_1-1} [1 - F(w)]^{m_2} [F(x) - F(w)]^{t-1-1} F(w) \, dw \]

\[ - (r+m_1-1) \int_w w^s [F(w)]^{r+m_1-2} [1 - F(w)]^{m_2} [F(x) - F(w)]^{t-1-1} F(w) \, dw \]  \hspace{1cm} (2.4)

by putting (2.4) in (2.3) and after simplification, we have the relation (2.1)

Theorem 2.2 For \( 1 \leq r < s < n - 1 \) and \( a, b, c \leq 1 \)

\[ B_1(i) \mu_{r+m_1+i+1, s-1}^{(a, b, c)} = \frac{\alpha \gamma(t-1)!}{m_2} \mu_{r+m_1+i+1, s-1}^{(a, b, c)} - \frac{r+m_1-1}{m_2} B_2(i) \mu_{r+m_1+i+1, s-1}^{(a, b, c)} \]

where

\[ B_1(i) = \sum_{i=0}^{m_2-1} \binom{m_2-1}{i} (r+m_1+i-1)! \]  \hspace{1cm} (2.5)

and

\[ B_2(i) = \sum_{i=0}^{m_2} \binom{m_2}{i} (r+m_1+i-2)! \]

\[ \gamma^{(a, b, c)}_{r+m_1+i, s-1} = C_{r, s, t, n} \int_{x} \int_{y} \gamma x^b y^c [1 - F(y)]^{s-1} I_t(x) f(x) f(y) \, dx \, dy \]  \hspace{1cm} (2.6)

having used (1.1), (1.3) and (1.5) where

\[ I_t(x) = \int_w \left[ w^s [F(w)]^{r+m_2} [1 - F(w)]^{m_2} \right]^{t-1} \, dw \]  \hspace{1cm} (2.7)

Integrating by parts, we have

\[ I_t(x) = m_2 \int_w w^s [F(w)]^{r+m_1-1} [1 - F(w)]^{m_2} f(w) \, dw \]

\[ - (r+m_1-1) \int_w w^s [F(w)]^{r+m_1-2} [1 - F(w)]^{m_2} f(w) \, dw \]  \hspace{1cm} (2.8)

substracting (2.8) into (2.6) and simplifying the resulting expression yields the relation (2.5).

Theorem 2.3 For \( 1 \leq r < s < t \leq n \) and \( a, b, c \geq 1 \)

\[ C_{r, s, t, n} \mu_{r+m_1+i+1, s-1}^{(a, b, c)} = \frac{\alpha \gamma b}{C_{r, s, t, n}(t+m_1-s-1)} \mu_{r+m_1+i+1, s-1}^{(a, b, c)} \]

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\[
\frac{(s + m_1 - r - 1)}{(t + m_2 - s - 1)} C_2(i, j) \mu_{r + m_1 - i + 1, s + 2m_2 - i - 1, t + m_2 - i + 1, n + 2m_2 - i - j - 1, n + 2m_2 - 2m_2 - t - j - 1}
\]

where

\[
C_2(i, j) = \sum_{i=0}^{m_1} \sum_{j=0}^{m_2} \left( \begin{array}{c} m_1 \\ i \end{array} \right) \left( \begin{array}{c} m_2 \\ j \end{array} \right) \frac{(r + m_1 - i - 1)! (s + m_1 - r - 1)!}{(n + 2m_2 + 2m_2 - i - j - 1)!} x(t + m_2 - s - 2)! (n + m_2 - t - j)!
\]

and

\[
\frac{(s + m_1 - r - 1)}{(t + m_2 - s - 1)} C_2(i, j) = \sum_{i=0}^{m_1} \sum_{j=0}^{m_2} \left( \begin{array}{c} m_1 \\ i \end{array} \right) \left( \begin{array}{c} m_2 \\ j \end{array} \right) \frac{(r + m_1 - i - 1)! (s + m_1 - r - 2)!}{(n + 2m_2 + 2m_2 - i - j - 1)!} x(t + m_2 - s - 1)! (n + m_2 - t - j)!
\]

\[
(2.9)
\]

**Proof**

\[
\gamma_{r, s, t, n} = C_{r, s, t, n} \int \int w^s x^b y^c [F(w)]^{r-1} [(1 - F(w))^{s-t}] K(w, y) f(w) f(y) dw dy
\]

(2.10)

where

\[
K(w, y) = \int [F(x) - F(w)]^{s-r-1} [F(y) - F(x)]^{s-t-1} [F(x)]^{m_1} [1 - F(x)]^{m_2} x^{b-1} dx
\]

(2.11)

having used (1.1), (1.3) and (1.5). Upon writing \(F(x) = F(x) - F(w) + F(w)\) and \(1 - F(x) = F(y) - F(x) + 1 - F(y)\) and using binomial expansion, we have

\[
K(w, y) = \sum_{i=0}^{m_1} \sum_{j=0}^{m_2} \left( \begin{array}{c} m_1 \\ i \end{array} \right) \left( \begin{array}{c} m_2 \\ j \end{array} \right) \int [F(w)]^{s+r-1} [F(y) - F(x)]^{r-m_2-s-1} x[F(w)]^{m_1} [1 - F(y)]^{m_2} x^{b-1} dx
\]

(2.12)

Integrating (2.12) by parts, we have

\[
K(w, y) = \frac{t + m_2 - s - 1}{b} \sum_{i=0}^{m_1} \sum_{j=0}^{m_2} \left( \begin{array}{c} m_1 \\ i \end{array} \right) \left( \begin{array}{c} m_2 \\ j \end{array} \right) \int [F(w)]^{m_1} [F(x) - F(w)]^{s+m_2-r-1} [F(y) - F(x)]^{r-m_2-s-1} x[F(y)]^{m_1} [1 - F(y)]^{m_2} f(x) dx
\]

\[
- \frac{s + m_1 - r - 1}{b} \sum_{i=0}^{m_1} \sum_{j=0}^{m_2} \left( \begin{array}{c} m_1 \\ i \end{array} \right) \left( \begin{array}{c} m_2 \\ j \end{array} \right) \int [F(w)]^{m_1} [F(x) - F(w)]^{s+m_2-r-1} [F(y) - F(x)]^{r-m_2-s-1} x[F(y)]^{m_1} [1 - F(y)]^{m_2} f(x) dx
\]

(2.13)

By putting the above expression into (2.10) and after simplification, we have the relation (2.9).

**Corollary 2.1** Setting \(s = r + 1, t = r + 2\) we have

\[
C_2(i, j) \mu_{r + m_1 - i + 1, s + 2m_2 - i - 1, t + m_2 - i + 1, n + 2m_2 - i - j - 1, n + 2m_2 - 2m_2 - t - j - 1} = \frac{\gamma b}{m_2} \mu_{r + m_1 - i + 1, s + 2m_2 - i - 1, t + m_2 - i + 1, n + 2m_2 - i - j - 1, n + 2m_2 - 2m_2 - t - j - 1}
\]

where

\[
C_2(i, j) = \sum_{i=0}^{m_1} \sum_{j=0}^{m_2} \left( \begin{array}{c} m_1 \\ i \end{array} \right) \left( \begin{array}{c} m_2 \\ j \end{array} \right) \frac{(r + m_1 - i + 1)! (m_1 - i)! (n + m_1 - r - j - 2)! n!}{(r-1)! (n-r-2)! (n + 2m_2 - i - j - 1)!}
\]

and

\[
C_2(i, j) = \sum_{i=0}^{m_1} \sum_{j=0}^{m_2} \left( \begin{array}{c} m_1 \\ i \end{array} \right) \left( \begin{array}{c} m_2 \\ j \end{array} \right) \frac{(r + m_1 - i + 1)! (m_1 - i)! (n + m_1 - r - j - 2)! n!}{(r-1)! (n-r-2)! (n + 2m_2 - i - j - 1)!}
\]

(2.13)

**Corollary 2.2** For \(s = r + 2\) and \(t = r + 1\)

\[
C_2(i, j) \mu_{r + m_1 - i + 1, s + 2m_2 - i - 1, t + m_2 - i + 1, n + 2m_2 - i - j - 1, n + 2m_2 - 2m_2 - t - j - 1} = \frac{\gamma b}{m_2} \mu_{r + m_1 - i + 1, s + 2m_2 - i - 1, t + m_2 - i + 1, n + 2m_2 - i - j - 1, n + 2m_2 - 2m_2 - t - j - 1}
\]

\[
\frac{(s + m_1 - r - 1)}{m_2} C_2(i, j) \mu_{r + m_1 - i + 1, s + 2m_2 - i - 1, t + m_2 - i + 1, n + 2m_2 - i - j - 1, n + 2m_2 - 2m_2 - t - j - 1}
\]

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where
\[
C_5(i,j) = \sum_{i=0}^{m_i} \sum_{j=0}^{m_j} \binom{m_i}{i} \binom{m_j}{j} \frac{(r+m_i-1)!(s+m_j-r-1)!}{(r-1)!(s-r-1)!(n-s-1)!(n+2m_i+2m_j-i-j-1)!} \times \frac{m_i}{n+m_i-s-j-1}! / n!
\]
and
\[
C_6(i,j) = \sum_{i=0}^{m_i} \sum_{j=0}^{m_j} \binom{m_i}{i} \binom{m_j}{j} \frac{(r+m_i-1)!(s+m_j-r-2)!}{(r-1)!(s-r-1)!(n-s-1)!(n+2m_i+2m_j-i-j-1)!} \times \frac{m_i}{n+m_i-s-j-1}! / n!
\]

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**Remark 2.1** In theorems 2.1, 2.2 and 2.3 if \(m_i = m_j = m\) we obtain relations for triple moments of order statistics from a symmetric generalized log-logistic distribution studied by Adeyemo[6].

**Remark 2.2** In theorems 2.1, 2.2 and 2.3 if \(m_i = m_j = 1\) we obtain relations for triple moments of order statistics from the ordinary log-logistic distribution studied by Ali and Kharri[6].

**Recurrence relations for quadruple moments**

**Theorem 3.1.** For \(1 \leq r < s < t < u \leq n\) and \(a, b, c, d \geq 1\)

\[
H_1(i,j,k) = \sum_{i=0}^{m_i} \sum_{j=0}^{m_j} \sum_{k=0}^{m_k} \binom{m_i}{i} \binom{m_j}{j} \binom{m_k}{k} \frac{(r+m_i-1)!(s+m_j-r-1)!(u+m_k-t-1)!(n+4m_i+4m_j+4m_k-3i-3j-k-3)}{(n+m_i+k-3i-2j)!} \times \frac{n+m_i-u}{u-t-1} \times \frac{m_i}{n+m_i-s-j-1}! / n!
\]

\[
H_2(i,j,k) = \sum_{i=0}^{m_i} \sum_{j=0}^{m_j} \sum_{k=0}^{m_k} \binom{m_i}{i} \binom{m_j}{j} \binom{m_k}{k} \frac{(r+m_i-1)!(s+m_j-r-1)!(u+m_k-t-1)!(n+4m_i+4m_j+4m_k-3i-3j-k-3)}{(n+m_i+k-3i-2j)!} \times \frac{n+m_i-u}{u-t-1} \times \frac{m_i}{n+m_i-s-j-1}! / n!
\]

\[
H_3(i,j,k) = \sum_{i=0}^{m_i} \sum_{j=0}^{m_j} \sum_{k=0}^{m_k} \binom{m_i}{i} \binom{m_j}{j} \binom{m_k}{k} \frac{(r+m_i-1)!(s+m_j-r-1)!(u+m_k-t-1)!(n+4m_i+4m_j+4m_k-3i-3j-k-3)}{(n+m_i+k-3i-2j)!} \times \frac{n+m_i-u}{u-t-1} \times \frac{m_i}{n+m_i-s-j-1}! / n!
\]

(3.1)
Proof

\[ \mathcal{Y}_{r, f, t, u}^{(a, b, c, d, n)} = C_{r, f, t, u} \int \int \int_{x, y} w^x y^y [F(w)]^{-1} [F(x) - F(w)]^{t-1} [F(y) - F(x)]^{s-1} x J_1(y) f(x) f(y) \, dx \, dy \]

where

\[ J_1(y) = \int z^{d-2} [F(z) - F(y)]^{t-1} [1 - F(z)]^{n-m_2-u} [F(z)]^{m_2} \, dz \]

having used (1.1), (1.4) and (1.6). Upon integrating (3.3) by parts writing \( F(z) = F(z) - F(y) + F(y) \), \( F(y) = F(x) - F(x) \) and \( F(x) = F(x) - F(w) + F(w) \) and using binomial expansion, we have

\[ J_1(y) = \frac{n + m_2 - u}{d-1} \sum_{i=0}^{m_2} \sum_{j=0}^{m_2} \sum_{k=0}^{m_2} \binom{m_2}{i} \binom{m_2}{j} \binom{m_2}{k} \]

\[ \times \int z^{d-1} [F(w)]^{m_2-i-j-k} [F(x) - F(w)]^{m_2-i-j-1} [F(y) - F(x)]^{m_2-i-j-1} \]

\[ \times [F(z) - F(y)]^{m_2-i-j-k} [1 - F(z)]^{n-m_2-u} f(z) \, dz \]

\[ - \frac{m_2}{d-1} \sum_{i=0}^{m_2} \sum_{j=0}^{m_2} \sum_{k=0}^{m_2} \binom{m_2}{i} \binom{m_2}{j} \binom{m_2}{k} \]

\[ \times \int z^{d-1} [F(w)]^{m_2-i-j-k-1} [F(x) - F(w)]^{m_2-i-j-1} [F(y) - F(x)]^{m_2-i-j-1} \]

\[ \times [F(z) - F(y)]^{m_2-i-j-k-2} [1 - F(z)]^{n-m_2-u} f(z) \, dz \]

\[ - \frac{u-t-1}{d-1} \sum_{i=0}^{m_2} \sum_{j=0}^{m_2} \sum_{k=0}^{m_2} \binom{m_2}{i} \binom{m_2}{j} \binom{m_2}{k} \]

\[ \times \int z^{d-1} [F(w)]^{m_2-i-j-k-1} [F(x) - F(w)]^{m_2-i-j-1} [F(y) - F(x)]^{m_2-i-j-1} \]

\[ \times [F(z) - F(y)]^{m_2-i-j-k-2} [1 - F(z)]^{n-m_2-u} f(z) \, dz \]

Upon substituting (3.4) into (3.2) and simplifying, we have the relation (3.1).

Theorem 3.2. For \( 1 \leq r < s < t < u \leq n \) and \( a, b, c, d \geq 1 \)

\[ H_{r, f, t, u}^{(a, b, c, d)} = \frac{t + m_2 - s - j - 1}{s - m_2 - r - 1} H_{r, f, t, u}^{(a, b, c, d)} \]

where

\[ H_{r, f, t, u}^{(a, b, c, d)} = \sum_{i=0}^{m_2} \sum_{j=0}^{m_2} \binom{m_2}{i} \binom{m_2}{j} \]

\[ \times \frac{(r + m_2 - i - 1)!(s + m_2 - r - 2)!(t + m_2 - s - j - 1)!}{(n + 2m_2 + 3m_2 - i - 2j)!} \]

\[ \times \frac{(u + m_2 - t - 1)!(n + m_2 - u - j)!}{(n + 2m_2 + 3m_2 - i - 2j)!} \]

\[ H_{r, f, t, u}^{(a, b, c, d)} = \sum_{i=0}^{m_2} \sum_{j=0}^{m_2} \binom{m_2}{i} \binom{m_2}{j} \]

\[ \times \frac{(r + m_2 - i - 1)!(s + m_2 - r - 1)!(t + m_2 - s - j - 2)!}{(n + 2m_2 + 3m_2 - i - 2j - 1)!} \]

\[ \times \frac{(u + m_2 - t - 1)!(n + m_2 - u - j)!}{(n + 2m_2 + 3m_2 - i - 2j - 1)!} \]

\[ \mathcal{Y}_{r, f, t, u}^{(a, b, c, d)} = C_{r, f, t, u} \]

(3.5)
Proof

\[ \gamma_{r,s,t,u}^{(a,b,c,d)} = C_{r,s,t,u} \int_{w} \int_{x} \int_{y} \int_{z} \left[ F(w)F^{-1}(z) - F(w)F^{-1}(y) \right]^{s-1} \times \left[ 1 - F(z)F^{-1}(w)F(y)F(z) \right]^{t-1} \times \left[ 1 - F(z)F^{-1}(w)F(y)F(z) \right]^{u-1} \, dx \]

(3.6)

where

\[ K(w,y) = \int_{w} x^{b-1} \left[ F(x) \right]^{m_0} \left[ 1 - F(x) \right]^{m_1} \left[ F(x) - F(w) \right]^{m_2} \left[ 1 - F(x) - F(y) \right]^{m_3} \, dx \]

having used (1.1), (1.4) and (1.6). Expressing \( 1 - F(x) \) as \( 1 - F(y) + F(y) - F(x) \) and \( 1 - F(y) \) as \( F(z) - F(y) + 1 - F(z) \), we have

\[ \gamma_{r,s,t,u}^{(a,b,c,d)} = C_{r,s,t,u} \sum_{i=0}^{m_0} \sum_{j=0}^{m_1} \left( \begin{array}{c} m_1 \\ i \\ j \end{array} \right) \gamma_{i,j}^{(a,b,c,d)} \]

(3.7)

By integrating (3.7) by parts, we obtain

\[ K(w,y) = \frac{t + s + 1 - s - j - 1}{b} \sum_{i=0}^{m_0} \sum_{j=0}^{m_1} \left( \begin{array}{c} m_1 \\ i \\ j \end{array} \right) \gamma_{i,j}^{(a,b,c,d)} \]

\[ \times \int_{x} x^{b-1} \left[ F(x) - F(w) \right]^{m_2} \left[ F(y) - F(x) \right]^{m_3} \, dx \]

- \frac{s + m_2 - r - 1}{b} \sum_{i=0}^{m_0} \sum_{j=0}^{m_1} \left( \begin{array}{c} m_1 \\ i \\ j \end{array} \right) \gamma_{i,j}^{(a,b,c,d)} \]

\[ \times \int_{x} x^{b-1} \left[ F(x) - F(w) \right]^{m_2} \left[ F(y) - F(x) \right]^{m_3} \, dx \]

(3.8)

By substituting (3.8) into (3.6) and simplifying the resulting expression, we obtain the relation (3.5).

**Corollary 3.1.** Setting \( s = r + 1 \), \( t = r + 2 \) and \( u = r + 3 \), we have

\[ H_{r+1,i,j}^{(a,b,c,d)} = \frac{m_2 - j}{m_1} H_{r,i,j}^{(a,b,c,d)} \]

\[ \times \frac{1}{(r - 1)!(n - r - 3)!} \gamma_{i,j}^{(a,b,c,d)} \]

Where

\[ H_{r,i,j}^{(a,b,c,d)} = \sum_{i=0}^{m_0} \sum_{j=0}^{m_1} \left( \begin{array}{c} m_1 \\ i \\ j \end{array} \right) \gamma_{i,j}^{(a,b,c,d)} \]

\[ \times \frac{(r + m_2 - i - 1)! (m_3 - i)! (m_4 - j)! (n + m_2 - r - j - 3)!}{(n + 2m_1 + 3m_2 - i - 2j)!} \]

(3.9)

**Corollary 3.2.** For \( s = r + 2 \) and \( u = s + 2 \), we have

\[ H_{r+2,i,j}^{(a,b,c,d)} = \frac{m_2 - j}{s + m_2 - r - 1} H_{r,i,j}^{(a,b,c,d)} \]

\[ \times \frac{1}{(r - 1)!(s - r - 1)!(n - s - 2)!} \gamma_{i,j}^{(a,b,c,d)} \]

\[ \times \frac{(r + m_2 - i - 1)! (m_3 - i)! (m_4 - j)! (n + m_2 - r - j - 3)!}{(n + 2m_1 + 3m_2 - i - 2j - 1)!} \]
where
\[
H_{x}(i,j) = \sum_{i=0}^{m_1} \sum_{j=0}^{m_2} \binom{m_1}{i} \binom{m_2}{j} \frac{(r + m_1 - i - 1)! (s + m_1 - r - 1)! (m_1 - j - 1)! m_2! (n + m_1 - s - j - 2)!}{(n + 2m_1 + 3m_2 - i - 2j - 1)!}
\]
\[
H_{y}(i,j) = \sum_{i=0}^{m_3} \sum_{j=0}^{m_4} \binom{m_3}{i} \binom{m_4}{j} \frac{(r + m_3 - i - 1)! (s + m_3 - r - 1)! (m_3 - j - 1)! m_4! (n + m_3 - s - j - 2)!}{(n + 2m_3 + 3m_4 - i - 2j - 1)!}
\]

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**Remark 3.1** In theorems 3.1 and 3.2 if we set \(m_1 = m_2 = m\) we obtain relations for quadruple moments of order statistics from a symmetric generalized log-logistic distribution studied by Adeyemi\[1\].

**Remark 3.1** In theorems 3.1 and 3.2 if we set \(m_3 = m_4 = 1\) we obtain relations for quadruple moments of order statistics from the ordinary log-logistic distribution studied by Ali and Khan\[2\].

**REFERENCES**


